

Canonical quantization of electromagnetic field in the presence of nonlinear anisotropic magnetodielectric medium with spatial-temporal dispersion

M. Amooshahi¹ *

¹ Faculty of science, University of Isfahan ,Hezar Jarib Ave., Isfahan,Iran

October 20, 2015

Abstract

Modeling a nonlinear anisotropic magnetodielectric medium with spatial-temporal dispersion by two continuum collections of three dimensional harmonic oscillators, a fully canonical quantization of the electromagnetic field is demonstrated in the presence of such a medium. Some coupling tensors of various ranks are introduced that couple the magnetodielectric medium with the electromagnetic field. The polarization and magnetization fields of the medium are defined in terms of the coupling tensors and the oscillators modeling the medium. The electric and magnetic susceptibility tensors of the medium are obtained in terms of the coupling tensors. It is shown that the electric field satisfy an integral equation in frequency domain. The integral equation is solved by an iteration method and the electric field is found up to an arbitrary accuracy.

PACS number 12.20.Ds

*amooshahi@sci.ui.ac.ir

1 Introduction

The quantum features of the electromagnetic field can be influenced by the magnetodielectric media. For example the Casimir effect [1] - [10] and the spontaneous emission rate of an initially excited atom [11]-[14] are modified in the presence of magnetodielectric media. There is a canonical quantization approach for the macroscopic electromagnetic field in a linear absorbing dielectric based on the damped polarization model[15]. In this scheme the electric polarization field of the medium is appeared in the Lagrangian of the total system as a part of the degrees of freedom of the medium. The other parts of the degrees of freedom of the absorbing dielectric are related to the dynamical variables of a reservoir coupled to the polarization field in order to inclusion the absorption property of the medium. The Hamiltonian of the total system is diagonalized in two steps. The first step is diagonalization of the polarization-reservoir part and the second step is diagonalization of the total Hamiltonian. This method has been generalized to inhomogeneous dielectrics [16]and anisotropic magnetodielectric media [17].

There is a new fully canonical quantization of the electromagnetic field in the presence of linear absorbing anisotropic magnetodielectric media in which the medium is modeled by two continuum collections of three dimensional space dependent harmonic oscillators[18],[19]. One of the collection describes the electric property and the another collection explains the magnetic property of the magnetodielectric medium. In contrast to the damped polarization model it is not needed the electric and magnetic polarization fields to be appeared in the Lagrangian of the total system. The two collections of the harmonic oscillators solely constitute the degrees of freedom of the medium. In fact the space dependent harmonic oscillators are able to describe both absorption and polarization properties of the medium. This means that in this approach the electric and magnetic polarization fields are defined in terms of the harmonic oscillators modeling the medium and the coupling tensors which couple the medium and electromagnetic field. Among the advantages of this approach is that this approach dose not need the processes of the diagonalization of the Hamiltonians of the different parts of the total system and is simply generalizable to the nonlinear anisotropic magnetodielectric medium with spatial-temporal dispersion. The aim of the present work is generalization of the mentioned scheme for a nonlinear anisotropic magnetodielectric medium with spatial-temporal dispersion. The canonical quantization of a three dimensional system in the presence of a nonlinear

anisotropic absorbing environment has been done previously and the effect of nonlinearity of the environment on the spontaneous emission of an initially excited two level atom has been investigated[20]

2 The Lagrangian of the total system

Modeling the nonlinear anisotropic magnetodielectric medium by two continuum collections three dimensional harmonic oscillators, the Lagrangian of the total system, that is the electromagnetic field plus the magnetodielectric medium, is introduced as the sum of three parts

$$L(t) = L_{em}(t) + L_{res}(t) + L_{int}(t) \quad (1)$$

The first part is the Lagrangian of the electromagnetic field which as usual can be written as

$$L_{em}(t) = \int d^3r \left[\frac{1}{2} \varepsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{\mathbf{B}^2(\mathbf{r}, t)}{2\mu_0} \right] \quad (2)$$

where $\mathbf{E} = -\vec{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ and \mathbf{A}, φ are respectively the vector and scalar potential which constitute the dynamical variables related to of the electromagnetic field. The second part in (1) is the Lagrangian of the magnetodielectric medium. If we denote the two continuum collections of three dimensional harmonic oscillators modeling the medium by $\mathbf{X}_\omega(\mathbf{r}, t)$ and $\mathbf{Y}_\omega(\mathbf{r}, t)$, with continuous parameter ω , then the Lagrangian of the medium is written as

$$\begin{aligned} L_{res}(t) = & \int d^3r \int_0^\infty d\omega \left[\frac{1}{2} \dot{\mathbf{X}}_\omega^2(\mathbf{r}, t) - \frac{1}{2} \omega^2 \mathbf{X}_\omega^2(\mathbf{r}, t) \right] \\ & + \int d^3r \int_0^\infty d\omega \left[\frac{1}{2} \dot{\mathbf{Y}}_\omega^2(\mathbf{r}, t) - \frac{1}{2} \omega^2 \mathbf{Y}_\omega^2(\mathbf{r}, t) \right] \end{aligned} \quad (3)$$

It will be shown that the two collections $\mathbf{X}_\omega(\mathbf{r}, t)$ and $\mathbf{Y}_\omega(\mathbf{r}, t)$ describe, respectively, the electric and magnetic properties of the medium. That is the electric polarization field of the medium is defined in terms of $\mathbf{X}_\omega(\mathbf{r}, t)$ and the magnetic polarization field of the medium is expressed in terms of $\mathbf{Y}_\omega(\mathbf{r}, t)$. The third term in (1) is the interaction part of the electromagnetic field and the magnetodielectric medium that for a nonlinear anisotropic medium with

spatial-temporal dispersion is proposed as

$$\begin{aligned}
L_{int}(t) = & \int_0^\infty d\omega_1 \int d^3r \int d^3r_1 f_{ii_1}^{(1)}(\omega_1, \mathbf{r}, \mathbf{r}_1) E^i(\mathbf{r}, t) X_{\omega_1}^{i_1}(\mathbf{r}_1, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3r \int d^3r_1 \int d^3r_2 [f_{ii_1 i_2}^{(2)}(\omega_1, \omega_2, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) \\
& \times E^i(\mathbf{r}, t) X_{\omega_1}^{i_1}(\mathbf{r}_1, t) X_{\omega_2}^{i_2}(\mathbf{r}_2, t)] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^\infty d\omega_3 \int d^3r \int d^3r_1 \int d^3r_2 \int d^3r_3 [f_{ii_1 i_2 i_3}^{(3)}(\omega_1, \omega_2, \omega_3, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
& \times E^i(\mathbf{r}, t) X_{\omega_1}^{i_1}(\mathbf{r}_1, t) X_{\omega_2}^{i_2}(\mathbf{r}_2, t) X_{\omega_3}^{i_3}(\mathbf{r}_3, t)] + \dots \\
& + \int_0^\infty d\omega_1 \int d^3r \int d^3r_1 g_{ii_1}^{(1)}(\omega_1, \mathbf{r}, \mathbf{r}_1) B^i(\mathbf{r}, t) Y_{\omega_1}^{i_1}(\mathbf{r}_1, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3r \int d^3r_1 \int d^3r_2 [g_{ii_1 i_2}^{(2)}(\omega_1, \omega_2, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) \\
& \times B^i(\mathbf{r}, t) Y_{\omega_1}^{i_1}(\mathbf{r}_1, t) Y_{\omega_2}^{i_2}(\mathbf{r}_2, t)] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^\infty d\omega_3 \int d^3r \int d^3r_1 \int d^3r_2 \int d^3r_3 [g_{ii_1 i_2 i_3}^{(3)}(\omega_1, \omega_2, \omega_3, \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
& \times B^i(\mathbf{r}, t) Y_{\omega_1}^{i_1}(\mathbf{r}_1, t) Y_{\omega_2}^{i_2}(\mathbf{r}_2, t) Y_{\omega_3}^{i_3}(\mathbf{r}_3, t)] + \dots
\end{aligned} \tag{4}$$

where the tensors $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \dots$ and $\mathbf{g}^{(1)}, \mathbf{g}^{(2)}, \mathbf{g}^{(3)}, \dots$ are the coupling tensors between the electromagnetic field and the nonlinear magnetodielectric medium. These coupling tensors play an important role in this quantization approach. It will be shown that the electric polarization field is obtained in terms of the coupling tensors $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \dots$ and the harmonic oscillators $\mathbf{X}_\omega(\mathbf{r}, t)$. Also the electric susceptibilities tensors of the medium are expressed in terms of the coupling tensors $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \dots$. Therefore these tensors are an indication for the ability of the polarization of the medium. Similarly the magnetic polarization field of the magnetodielectric medium is defined in terms of the coupling tensors $\mathbf{g}^{(1)}, \mathbf{g}^{(2)}, \mathbf{g}^{(3)}, \dots$ and the harmonic oscillators $\mathbf{Y}_\omega(\mathbf{r}, t)$. As well the magnetic susceptibility tensors of the medium are written in terms of the coupling tensors $\mathbf{g}^{(1)}, \mathbf{g}^{(2)}, \mathbf{g}^{(3)}, \dots$ and therefore these tensors explain the ability of magnetization of the medium. The Lagrangian (1)-(4) is a nonlocal one and there is some problems for a canonical quantization using this Lagrangian. The easiest way to overcome the difficulties related to a nonlocal Lagrangian is to work in the reciprocal space. Therefore we write the classical fields and the coupling tensors appeared in the

Lagrangian (1)-(4) in terms of their spatial fourier transforms. For example for the field $\mathbf{X}_\omega(\mathbf{r}, t)$ we write

$$\mathbf{X}_\omega(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \underline{\mathbf{X}}_\omega(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{r}} \quad (5)$$

Similarly the components of the coupling tensors $\mathbf{f}^{(n)}$ and $\mathbf{g}^{(n)}$ can be written in terms of their Fourier transforms as

$$\begin{aligned} & f_{ii_1 i_2 \dots i_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_n, \mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_n) \\ &= \frac{1}{\frac{3(n+1)}{2}} \int d^3k \int d^3k_1 \dots \int d^3k_n \underline{f}_{ii_1 \dots i_n}^{(n)}(\omega_1, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n) e^{i\mathbf{k} \cdot \mathbf{r} - i\mathbf{k}_1 \cdot \mathbf{r}_1 - \dots - i\mathbf{k}_n \cdot \mathbf{r}_n} \\ & g_{ii_1 i_2 \dots i_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_n, \mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_n) \\ &= \frac{1}{\frac{3(n+1)}{2}} \int d^3k \int d^3k_1 \dots \int d^3k_n \underline{g}_{ii_1 \dots i_n}^{(n)}(\omega_1, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n) e^{i\mathbf{k} \cdot \mathbf{r} - i\mathbf{k}_1 \cdot \mathbf{r}_1 - \dots - i\mathbf{k}_n \cdot \mathbf{r}_n} \end{aligned} \quad (6)$$

Since the fields appeared in Lagrangian (1)-(4) are real valued, we have

$$\underline{\mathbf{X}}_\omega(\mathbf{k}, t) = \underline{\mathbf{X}}_\omega^*(\mathbf{k}, t) \quad (7)$$

for the field $\mathbf{X}_\omega(\mathbf{r}, t)$ and the other fields in the Lagrangian (1)-(4). Similar relations for the real valued coupling tensors $\mathbf{f}^{(n)}$ and $\mathbf{g}^{(n)}$ are as follows

$$\begin{aligned} \underline{f}_{ii_1 \dots i_n}^{(n)}(\omega_1, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n) &= (\underline{f}_{ii_1 \dots i_n}^{(n)})^*(\omega_1, \dots, \omega_n, -\mathbf{k}, -\mathbf{k}_1, \dots, -\mathbf{k}_n) \\ \underline{g}_{ii_1 \dots i_n}^{(n)}(\omega_1, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n) &= (\underline{g}_{ii_1 \dots i_n}^{(n)})^*(\omega_1, \dots, \omega_n, -\mathbf{k}, -\mathbf{k}_1, \dots, -\mathbf{k}_n) \end{aligned} \quad (8)$$

Because of the relations (7) and (8) the independent dynamical variables can be taken into account by restricting the integration domain to the half space $k_z \geq 0$ in the reciprocal space. Substituting the real valued fields and the coupling tensors $\mathbf{f}^{(n)}$ and $\mathbf{g}^{(n)}$ in the total Lagrangian (1)-(4) by their Fourier transforms and using the relation (7) for $\mathbf{X}_\omega(\mathbf{r}, t)$, and similar relations for the other fields and also using (8), one can rewrite the Lagrangian of the total

system as

$$\begin{aligned}
L(t) = & \int' d^3k \left[\varepsilon_0 |\dot{\underline{\mathbf{A}}}(\mathbf{k}, t)|^2 + \varepsilon_0 k^2 |\underline{\varphi}(\mathbf{k}, t)|^2 - \frac{|\mathbf{k} \times \underline{\mathbf{A}}(\mathbf{k}, t)|^2}{\mu_0} \right] \\
& + \varepsilon_0 \int' d^3k \left[-i\mathbf{k} \cdot \dot{\underline{\mathbf{A}}}(\mathbf{k}, t) \underline{\varphi}^*(\mathbf{k}, t) + H.C \right] \\
& + \int_0^\infty d\omega \int' d^3k \left[|\dot{\underline{\mathbf{X}}}_\omega(\mathbf{k}, t)|^2 - \omega^2 |\underline{\mathbf{X}}_\omega(\mathbf{k}, t)|^2 \right] \\
& + \int_0^\infty d\omega \int' d^3k \left[|\dot{\underline{\mathbf{Y}}}_\omega(\mathbf{k}, t)|^2 - \omega^2 |\underline{\mathbf{Y}}_\omega(\mathbf{k}, t)|^2 \right] \\
& + \int_0^\infty d\omega_1 \int' d^3k \int' d^3k_1 \left[\underline{f}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{E}^{*i}(\mathbf{k}, t) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) + H.C \right] \\
& + \int_0^\infty d\omega_1 \int' d^3k \int' d^3k_1 \left[\underline{f}_{ij}^{(1)}(\omega_1, -\mathbf{k}, \mathbf{k}_1) \underline{E}^i(\mathbf{k}, t) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) + H.C \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \right. \\
& \times \underline{E}^{*i}(\mathbf{k}, t) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) \underline{X}_{\omega_2}^k(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, -\mathbf{k}_2) \right. \\
& \times \underline{E}^{*i}(\mathbf{k}, t) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) \underline{X}_{\omega_2}^k(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, -\mathbf{k}_1, \mathbf{k}_2) \right. \\
& \times \underline{E}^{*i}(\mathbf{k}, t) \underline{X}_{\omega_1}^{*j}(\mathbf{k}_1, t) \underline{X}_{\omega_2}^k(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2) \right. \\
& \times \underline{E}^{*i}(\mathbf{k}, t) \underline{X}_{\omega_1}^{*j}(\mathbf{k}_1, t) \underline{X}_{\omega_2}^{*k}(\mathbf{k}_2, t) + H.C \left. \right] + \dots \\
& + \int_0^\infty d\omega_1 \int' d^3k \int' d^3k_1 \left[\underline{g}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{B}^{*i}(\mathbf{k}, t) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) + H.C \right] \\
& + \int_0^\infty d\omega_1 \int' d^3k \int' d^3k_1 \left[\underline{g}_{ij}^{(1)}(\omega_1, -\mathbf{k}, \mathbf{k}_1) \underline{B}^i(\mathbf{k}, t) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) + H.C \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \right. \\
& \times \underline{B}^{*i}(\mathbf{k}, t) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) \underline{Y}_{\omega_2}^k(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, -\mathbf{k}_2) \right. \\
& \times \underline{B}^{*i}(\mathbf{k}, t) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) \underline{Y}_{\omega_2}^{*k}(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, -\mathbf{k}_1, \mathbf{k}_2) \right. \\
& \times \underline{B}^{*i}(\mathbf{k}, t) \underline{Y}_{\omega_1}^{*j}(\mathbf{k}_1, t) \underline{Y}_{\omega_2}^k(\mathbf{k}_2, t) + H.C \left. \right] \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int' d^3k \int' d^3k_1 \int' d^3k_2 \left[\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2) \right.
\end{aligned}$$

where $\int' d^3k$ denote the integration over the half space $k_z \geq 0$ and $\underline{\mathbf{E}}(\mathbf{k}, t) = -\dot{\underline{\mathbf{A}}}(\mathbf{k}, t) - i\mathbf{k}\underline{\varphi}(\mathbf{k}, t)$ and $\underline{\mathbf{B}}(\mathbf{k}, t) = i\mathbf{k} \times \underline{\mathbf{A}}(\mathbf{k}, t)$. In the Lagrangian (9) the three points ... indicate the other terms which are dependent on the coupling tensors more than the third rank. Hereafter we apply the symbols $\int' d^3k$ and $\int d^3k$ for the integration over the half space $k_z \geq 0$ and integration over the total reciprocal space, respectively.

3 Classical Euler-Lagrange equations

The Lagrangian (9) does not involve the space derivatives of the dynamical variables and one can easily obtain the classical Euler-Lagrange equations for the Fourier transforms of the fields appeared in the Lagrangian of the total system. The classical Euler-Lagrange equation for the scalar potential $\underline{\varphi}(\mathbf{k}, t)$ and vector potential $\underline{\mathbf{A}}(\mathbf{k}, t)$ are respectively as follows

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\underline{\varphi}}^*(\mathbf{k}, t)} \right) - \frac{\delta L}{\delta \underline{\varphi}^*(\mathbf{k}, t)} &= 0 \\ \Rightarrow -\varepsilon_0 i\mathbf{k} \cdot \dot{\underline{\mathbf{A}}}(\mathbf{k}, t) + \varepsilon_0 k^2 \underline{\varphi}(\mathbf{k}, t) &= -i\mathbf{k} \cdot \underline{\mathbf{P}}(\mathbf{k}, t) \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\underline{A}}_i^*(\mathbf{k}, t)} \right) - \frac{\delta L}{\delta \underline{A}_i^*(\mathbf{k}, t)} &= 0 \quad i = 1, 2, 3 \\ \Rightarrow \mu_0 \varepsilon_0 \ddot{\underline{\mathbf{A}}}(\mathbf{k}, t) + \mu_0 \varepsilon_0 i\mathbf{k} \dot{\underline{\varphi}}(\mathbf{k}, t) - \mathbf{k} \times (\mathbf{k} \times \underline{\mathbf{A}}(\mathbf{k}, t)) \\ &= \mu_0 \dot{\underline{\mathbf{P}}}(\mathbf{k}, t) + i\mu_0 \mathbf{k} \times \underline{\mathbf{M}}(\mathbf{k}, t) \end{aligned} \quad (11)$$

where $\underline{\mathbf{P}}(\mathbf{k}, t)$ and $\underline{\mathbf{M}}(\mathbf{k}, t)$ are the Fourier transforms of the electric and magnetic polarization densities of the nonlinear magnetodielectric medium, respectively. The components of the polarization densities are defined in terms of the coupling tensors $\underline{\mathbf{f}}^{(n)}, \underline{\mathbf{g}}^{(n)}$ and the harmonic oscillators modeling

the medium as

$$\begin{aligned}
\underline{P}_i(\mathbf{k}, t) = & \int_0^\infty d\omega_1 \int d^3k_1 \underline{f}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3k_1 \int d^3k_2 \underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) \underline{X}_{\omega_2}^k(\mathbf{k}_2, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^\infty d\omega_3 \int d^3k_1 \int d^3k_2 \int d^3k_3 \underline{f}_{ijkl}^{(3)}(\omega_1, \omega_2, \omega_3, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\
& \times \underline{X}_{\omega_1}^j(\mathbf{k}_1, t) \underline{X}_{\omega_2}^k(\mathbf{k}_2, t) \underline{X}_{\omega_3}^l(\mathbf{k}_3, t) + \dots
\end{aligned} \tag{12}$$

$$\begin{aligned}
\underline{M}_i(\mathbf{k}, t) = & \int_0^\infty d\omega_1 \int d^3k_1 \underline{g}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3k_1 \int d^3k_2 \underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) \underline{Y}_{\omega_2}^k(\mathbf{k}_2, t) \\
& + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^\infty d\omega_3 \int d^3k_1 \int d^3k_2 \int d^3k_3 \underline{g}_{ijkl}^{(3)}(\omega_1, \omega_2, \omega_3, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\
& \times \underline{Y}_{\omega_1}^j(\mathbf{k}_1, t) \underline{Y}_{\omega_2}^k(\mathbf{k}_2, t) \underline{Y}_{\omega_3}^l(\mathbf{k}_3, t) + \dots
\end{aligned} \tag{13}$$

4 Canonical quantization

In order to quantize the electromagnetic field canonically, one should be eliminated the extra degrees of freedom from the total Lagrangian (9). For this purpose we apply the coulomb gauge and eliminate the scalar potential $\varphi(\mathbf{k}, t)$ from the total Lagrangian using the equation (10). In reciprocal space the coulomb gauge is in the form $\mathbf{k} \cdot \underline{\mathbf{A}} = 0$. Applying the coulomb gauge and using the equation (10) one can obtain the spatial Fourier transform of the scalar potential as

$$\underline{\varphi}(\vec{k}, t) = -\frac{i\vec{k} \cdot \underline{\vec{P}}(\vec{k}, t)}{\varepsilon_0 k^2} \tag{14}$$

Let $\mathbf{e}_{\mathbf{k}\lambda}$, $\lambda = 1, 2$ and $\mathbf{e}_{\mathbf{k}3} = \hat{k} = \frac{\mathbf{k}}{k}$ be three mutual orthogonal unit vectors for each vector \mathbf{k} . Then from the coulomb gauge $\mathbf{k} \cdot \underline{\mathbf{A}} = 0$, we are admissible expand the vector potential along the orthogonal unit vector $\mathbf{e}_{\mathbf{k}\lambda}$, $\lambda = 1, 2$ as

$$\underline{\vec{A}}(\vec{k}, t) = \sum_{\lambda=1}^2 \underline{A}_\lambda(\vec{k}, t) \vec{e}_{\lambda\vec{k}} \tag{15}$$

where $\underline{A}_\lambda(\vec{k}, t)$, $\lambda = 1, 2$ are the new dynamical variables of electromagnetic field. Also we introduce the new dynamical variables $\underline{X}_{\omega\lambda}(\mathbf{k}, t)$ and $\underline{Y}_{\omega\lambda}(\mathbf{k}, t)$ for the nonlinear magnetodielectric medium as the coefficients of the expansion of the fields $\underline{\mathbf{X}}_\omega(\mathbf{k}, t)$ and $\underline{\mathbf{Y}}_\omega(\mathbf{k}, t)$ in terms of the orthogonal unit vectors $\mathbf{e}_{\mathbf{k}\lambda}$, $\lambda = 1, 2, 3$, that is

$$\begin{aligned}\vec{X}_\omega(\vec{k}, t) &= \sum_{\lambda=1}^3 \underline{X}_{\omega\lambda}(\vec{k}, t) \vec{e}_{\lambda\vec{k}} \\ \vec{Y}_\omega(\vec{k}, t) &= \sum_{\lambda=1}^3 \underline{Y}_{\omega\lambda}(\vec{k}, t) \vec{e}_{\lambda\vec{k}}\end{aligned}\quad (16)$$

Substituting the scalar potential $\varphi(\mathbf{k}, t)$ from (14) in the Lagrangian (9) and using the expansions (15) and (16), one can rewrite the Lagrangian of the total system in terms of the new coordinates of the system as

$$\begin{aligned}L(t) &= \int_0^\infty d\omega \int' d^3k \sum_{\lambda=1}^3 \left(|\dot{\underline{X}}_{\omega\lambda}|^2 - \omega^2 |\underline{X}_{\omega\lambda}|^2 + |\dot{\underline{Y}}_{\omega\lambda}|^2 - \omega^2 |\underline{Y}_{\omega\lambda}|^2 \right) \\ &+ \int' d^3k \sum_{\lambda=1}^2 \left(\varepsilon_0 |\dot{\underline{A}}_\lambda|^2 - \frac{k^2 |\underline{A}_\lambda|^2}{\mu_0} \right) - \frac{1}{\varepsilon_0} \int' d^3k \frac{|\mathbf{k} \cdot \underline{\mathbf{P}}|^2}{k^2} \\ &+ \int' d^3k \left[\left(- \sum_{\lambda=1}^2 \dot{\underline{A}}_\lambda \mathbf{e}_{\lambda\mathbf{k}} \right) \cdot \underline{\mathbf{P}}^*(\mathbf{k}, t) + \left(i\mathbf{k} \times \sum_{\lambda=1}^2 \underline{A}_\lambda \mathbf{e}_{\lambda\mathbf{k}} \right) \cdot \underline{\mathbf{M}}^*(\mathbf{k}, t) + H.c \right]\end{aligned}\quad (17)$$

where the polarization densities $\underline{\mathbf{P}}(\mathbf{k}, t)$, $\underline{\mathbf{M}}(\mathbf{k}, t)$ have previously been defined in the relations (12) and (13). The relation (17) give us the Lagrangian of the total system in which the extra degrees of freedom have been eliminated and can be used for a canonical quantization of the electromagnetic field together with the nonlinear anisotropic magnetodielectric medium. The canonical conjugate momenta of the total system for any vector \mathbf{k} in the half space

$k_z \geq 0$ can be defined as

$$\begin{aligned}
-\underline{D}_\lambda(\vec{k}, t) &= \frac{\delta \underline{L}}{\delta \left(\dot{\underline{A}}_\lambda^*(\vec{k}, t) \right)} = \varepsilon_0 \dot{\underline{A}}_\lambda(\mathbf{k}, t) - \mathbf{e}_{\lambda \mathbf{k}} \cdot \underline{\mathbf{P}}(\vec{k}, t) \\
\underline{Q}_{\omega\lambda}(\vec{k}, t) &= \frac{\delta \underline{L}}{\delta \left(\dot{\underline{X}}_{\omega\lambda}^*(\vec{k}, t) \right)} = \dot{\underline{X}}_{\omega\lambda}(\vec{k}, t) \\
\underline{\Pi}_{\omega\lambda}(\vec{k}, t) &= \frac{\delta \underline{L}}{\delta \left(\dot{\underline{Y}}_{\omega\lambda}^*(\vec{k}, t) \right)} = \dot{\underline{Y}}_{\omega\lambda}(\vec{k}, t)
\end{aligned} \tag{18}$$

Now, following the standard way for a canonical quantization, we impose the following commutation relations between the conjugate dynamical variables of the total system

$$\begin{aligned}
\left[\underline{A}_\lambda^*(\vec{k}, t), -\underline{D}_{\lambda'}(\vec{k}', t) \right] &= i\hbar \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \\
\left[\underline{X}_{\omega\lambda}^*(\vec{k}, t), \underline{Q}_{\omega'\lambda'}(\vec{k}', t) \right] &= i\hbar \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\vec{k} - \vec{k}') \\
\left[\underline{Y}_{\omega\lambda}^*(\vec{k}, t), \underline{\Pi}_{\omega'\lambda'}(\vec{k}', t) \right] &= i\hbar \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\vec{k} - \vec{k}')
\end{aligned} \tag{19}$$

The Hamiltonian of the total system can be written as usual way as

$$\begin{aligned}
H(t) &= \int' d^3k \sum_{\lambda=1}^2 \left[-\underline{D}_\lambda \dot{\underline{A}}_\lambda^* - \underline{D}_\lambda^* \dot{\underline{A}}_\lambda \right] \\
&+ \int_0^\infty d\omega \int' d^3k \sum_{\lambda=1}^2 \left[\underline{Q}_{\omega\lambda} \dot{\underline{X}}_{\omega\lambda}^* + \underline{Q}_{\omega\lambda}^* \dot{\underline{X}}_{\omega\lambda} \right] \\
&+ \int_0^\infty d\omega \int' d^3k \sum_{\lambda=1}^2 \left[\underline{\Pi}_{\omega\lambda} \dot{\underline{Y}}_{\omega\lambda}^* + \underline{\Pi}_{\omega\lambda}^* \dot{\underline{Y}}_{\omega\lambda} \right] - L(t)
\end{aligned} \tag{20}$$

Using the definitions of conjugate momenta given in (18), the Hamiltonian of the system in terms of the coordinates of the system and their conjugate

momenta becomes

$$\begin{aligned}
H(t) = & \int' d^3k \sum_{\lambda=1}^2 \left(\frac{|\underline{D}_\lambda - \mathbf{e}_{\lambda\mathbf{k}} \cdot \underline{\mathbf{P}}|^2}{\varepsilon_0} + \frac{k^2 |\underline{A}_\lambda|^2}{\mu_0} \right) + \frac{1}{\varepsilon_0} \int' d^3k \frac{|\mathbf{k} \cdot \underline{\mathbf{P}}|^2}{k^2} \\
& - \int' d^3k \left[\left(i\mathbf{k} \times \sum_{\lambda=1}^2 \underline{A}_\lambda \mathbf{e}_{\lambda\mathbf{k}} \right) \cdot \underline{\mathbf{M}}^* + H.c \right] \\
& + \int_0^\infty d\omega \int' d^3k \sum_{\lambda=1}^3 \left(|\underline{Q}_{\omega\lambda}|^2 + \omega^2 |\underline{X}_{\omega\lambda}|^2 + |\underline{\Pi}_{\omega\lambda}|^2 + \omega^2 |\underline{Y}_{\omega\lambda}|^2 \right)
\end{aligned} \tag{21}$$

It can be easily shown that the Heisenberg equation for $\underline{A}_\lambda(\mathbf{k}, t)$, $\lambda = 1, 2$ leads to the relation $\underline{\mathbf{D}}(\mathbf{k}, t) = \varepsilon_0 \underline{\mathbf{E}}(\mathbf{k}, t) + \underline{\mathbf{P}}(\mathbf{k}, t)$ [18], where $\underline{\mathbf{D}} = \sum_{\lambda=1}^2 \underline{D}_\lambda \mathbf{e}_{\lambda\mathbf{k}}$ is the displacement field and $\underline{\mathbf{E}} = -\dot{\underline{\mathbf{A}}} - \frac{\mathbf{k}(\mathbf{k} \cdot \underline{\mathbf{P}})}{\varepsilon_0}$ is the Fourier transform of the electric field. Also the Heisenberg equation for \underline{D}_λ , $\lambda = 1, 2$ give us the Maxwell equation $\dot{\underline{\mathbf{D}}}(\mathbf{k}, t) = i\mathbf{k} \times \underline{\mathbf{H}}(\mathbf{k}, t)$ in the reciprocal space, where $\underline{\mathbf{H}} = \frac{i\mathbf{k} \times \underline{\mathbf{A}}}{\mu_0} - \underline{\mathbf{M}}$ is the magnetic induction field [18].

5 The constitutive relations of the medium

One can obtain the constitutive relations of the nonlinear magnetodielectric medium using the definitions of polarization and magnetization densities of the medium given by (12),(13) and the Heisenberg equations for the dynamical variables $\underline{X}_{\omega\lambda}(\mathbf{k}, t)$, $\underline{Y}_{\omega\lambda}(\mathbf{k}, t)$, $\lambda = 1, 2, 3$. Straightforwardly, one can show that the combination of the Heisenberg equations of the conjugate variables $\underline{X}_{\omega\lambda}(\mathbf{k}, t)$, $\underline{Q}_{\omega\lambda}(\mathbf{k}, t)$, $\lambda = 1, 2, 3$ leads to the following equation of motion for the components of the field $\underline{\mathbf{X}}(\mathbf{k}, t)$

$$\begin{aligned}
\ddot{\underline{X}}_{\omega i}(\mathbf{k}, t) + \omega^2 \underline{X}_{\omega i}(\mathbf{k}, t) = & \int d^3k_1 \underline{f}_{ij}^{\dagger(1)}(\omega, \mathbf{k}_1, \mathbf{k}) \underline{E}^j(\mathbf{k}_1, t) \\
& + \int_0^\infty d\omega_2 \int d^3k_1 \int d^3k_2 \underline{f}_{jik}^{*(2)}(\omega, \omega_2, \mathbf{k}_1, \mathbf{k}, \mathbf{k}_2) \underline{E}^j(\mathbf{k}_1, t) \underline{X}^{*k}(\mathbf{k}_2, t) \\
& + \int_0^\infty d\omega_1 \int d^3k_1 \int d^3k_2 \underline{f}_{kji}^{*(2)}(\omega_1, \omega, \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}) \underline{E}^k(\mathbf{k}_2, t) \underline{X}^{*j}(\mathbf{k}_1, t) + \dots
\end{aligned} \tag{22}$$

where the completeness relation $\sum_{\lambda=1}^3 \mathbf{e}_{\mathbf{k}\lambda i} \mathbf{e}_{\mathbf{k}\lambda j} = \delta_{ij}$ has been used, three points ... indicate the terms containing the coupling tensors more than the third rank and $\underline{\mathbf{E}} = -\underline{\dot{\mathbf{A}}} - \frac{\mathbf{k}(\mathbf{k} \cdot \underline{\mathbf{P}})}{\varepsilon_0}$ is the Fourier transform of the electric field. Similarly the combination of the Heisenberg equations of the conjugate variables $\underline{Y}_{\omega\lambda}, \underline{\Pi}_{\omega\lambda}, \lambda = 1, 2, 3$ give us

$$\begin{aligned} \ddot{\underline{Y}}_{\omega i}(\mathbf{k}, t) + \omega^2 \underline{Y}_{\omega i}(\mathbf{k}, t) &= \int d^3 k_1 \underline{g}_{ij}^{\dagger(1)}(\omega, \mathbf{k}_1, \mathbf{k}) \underline{B}^j(\mathbf{k}_1, t) \\ &+ \int_0^\infty d\omega_2 \int d^3 k_1 \int d^3 k_2 \underline{g}_{jik}^{*(2)}(\omega, \omega_2, \mathbf{k}_1, \mathbf{k}, \mathbf{k}_2) \underline{B}^j(\mathbf{k}_1, t) \underline{Y}^{*k}(\mathbf{k}_2, t) \\ &+ \int_0^\infty d\omega_1 \int d^3 k_1 \int d^3 k_2 \underline{g}_{kji}^{*(2)}(\omega_1, \omega, \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}) \underline{B}^k(\mathbf{k}_2, t) \underline{Y}^{*j}(\mathbf{k}_1, t) + \dots \end{aligned} \quad (23)$$

where $\underline{\mathbf{B}}(\mathbf{k}, t) = i\mathbf{k} \times \underline{\mathbf{A}}(\mathbf{k}, t)$ is the Fourier transform of the magnetic field. In (22) and (23) the components of the tensors $\underline{\mathbf{f}}^{\dagger(1)}$ and $\underline{\mathbf{g}}^{\dagger(1)}$ are defined as $\underline{f}_{ij}^{\dagger(1)} = \underline{f}_{ji}^{*(1)}, \underline{g}_{ij}^{\dagger(1)} = \underline{g}_{ji}^{*(1)}$. The differential equations (22) and (23) are two frames of the complicated nonlinear coupled differential equations. The solution of these equations exactly is impossible unless an iteration method is used. One may apply the first order approximation and keep only the first term in the right hand of the equations (22) and (23) and write the solutions of the differential equations (22) and (23) as

$$\underline{X}_{\omega i}(\mathbf{k}, t) = \underline{X}_{N\omega i}(\mathbf{k}, t) + \int_{-\infty}^t dt' \frac{\sin \omega(t-t')}{\omega} \int d^3 k_1 \underline{f}_{ij}^{\dagger(1)}(\omega, \mathbf{k}_1, \mathbf{k}) \underline{E}^j(\mathbf{k}_1, t') \quad (24)$$

$$\underline{Y}_{\omega i}(\mathbf{k}, t) = \underline{Y}_{N\omega i}(\mathbf{k}, t) + \int_{-\infty}^t dt' \frac{\sin \omega(t-t')}{\omega} \int d^3 k_1 \underline{g}_{ij}^{\dagger(1)}(\omega, \mathbf{k}_1, \mathbf{k}) \underline{B}^j(\mathbf{k}_1, t') \quad (25)$$

where $\underline{\mathbf{X}}_{N\omega}(\mathbf{k}, t)$ and $\underline{\mathbf{Y}}_{N\omega}(\mathbf{k}, t)$ are the solutions of the homogenous equations

$$\begin{aligned} \ddot{\underline{\mathbf{X}}}_{N\omega}(\mathbf{k}, t) + \omega^2 \underline{\mathbf{X}}_{N\omega}(\mathbf{k}, t) &= 0 \\ \ddot{\underline{\mathbf{Y}}}_{N\omega}(\mathbf{k}, t) + \omega^2 \underline{\mathbf{Y}}_{N\omega}(\mathbf{k}, t) &= 0 \end{aligned} \quad (26)$$

respectively which can be written as

$$\begin{aligned}\underline{\mathbf{X}}_{N\omega}(\mathbf{k}, t) &= \sqrt{\frac{\hbar}{2\omega}} \sum_{\lambda=1}^3 [b_{\lambda}(\omega, \mathbf{k})e^{-i\omega t} + b_{\lambda}^{\dagger}(\omega, \mathbf{k})e^{i\omega t}] \mathbf{e}_{\mathbf{k}\lambda} \\ \underline{\mathbf{Y}}_{N\omega}(\mathbf{k}, t) &= \sqrt{\frac{\hbar}{2\omega}} \sum_{\lambda=1}^3 [d_{\lambda}(\omega, \mathbf{k})e^{-i\omega t} + d_{\lambda}^{\dagger}(\omega, \mathbf{k})e^{i\omega t}] \mathbf{e}_{\mathbf{k}\lambda}\end{aligned}\quad (27)$$

where the operators $b_{\lambda}(\omega, \mathbf{k})$, $b_{\lambda}^{\dagger}(\omega, \mathbf{k})$ and $d_{\lambda}(\omega, \mathbf{k})$, $d_{\lambda}^{\dagger}(\omega, \mathbf{k})$ satisfy the commutation relations

$$\begin{aligned}[b_{\lambda}(\omega, \mathbf{k}), b_{\lambda'}^{\dagger}(\omega', \mathbf{k}')] &= \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}') \\ [d_{\lambda}(\omega, \mathbf{k}), d_{\lambda'}^{\dagger}(\omega', \mathbf{k}')] &= \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}')\end{aligned}\quad (28)$$

in compatible to the commutation relations (19).

Now substituting $\underline{\mathbf{X}}_{\omega}(\mathbf{k}, t)$ from (24) in (12) and $\underline{\mathbf{Y}}_{\omega}(\mathbf{k}, t)$ from (25) in (13) give us the constitutive relations of the nonlinear anisotropic magnetodielectric medium as

$$\begin{aligned}\underline{P}_i(\mathbf{k}, t) &= \underline{P}_{Ni}(\mathbf{k}, t) + \int_{-\infty}^{+\infty} dt_1 \int d^3k_1 \chi_{ij}^{(1)}(t - t_1, \mathbf{k}, \mathbf{k}_1) \underline{E}^j(\mathbf{k}_1, t_1) \\ &+ \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \int d^3k_1 \int d^3k_2 \chi_{ijk}^{(2)}(t - t_1, t - t_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{E}^j(\mathbf{k}_1, t_1) \underline{E}^k(\mathbf{k}_2, t_2) \\ &+ \dots\end{aligned}\quad (29)$$

$$\begin{aligned}\underline{M}_i(\mathbf{k}, t) &= \underline{M}_{Ni}(\mathbf{k}, t) + \int_{-\infty}^{+\infty} dt_1 \int d^3k_1 \zeta_{ij}^{(1)}(t - t_1, \mathbf{k}, \mathbf{k}_1) \underline{B}^j(\mathbf{k}_1, t_1) \\ &+ \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \int d^3k_1 \int d^3k_2 \zeta_{ijk}^{(2)}(t - t_1, t - t_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{B}^j(\mathbf{k}_1, t_1) \underline{B}^k(\mathbf{k}_2, t_2) \\ &+ \dots\end{aligned}\quad (30)$$

where the tensors χ^n and ζ^n for $n = 1, 2, \dots$ are respectively the electric and magnetic susceptibility tensors and define in terms of the coupling tensors

$\underline{\mathbf{f}}^{(n)}$ and $\underline{\mathbf{g}}^{(n)}$ as

$$\begin{aligned}
& \chi_{ii_1 i_2 \dots i_n}^{(n)}(t_1, t_2, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n) = \\
& \Theta(t_1) \Theta(t_2) \dots \Theta(t_n) \int_0^\infty d\omega_1 \int_0^d \omega_2 \dots \int_0^\infty d\omega_n \frac{\sin \omega_1 t_1}{\omega_1} \frac{\sin \omega_2 t_2}{\omega_2} \dots \frac{\sin \omega_n t_n}{\omega_n} \\
& \times \int d^3 p_1 \int d^3 p_2 \dots \int d^3 p_n [\underline{f}_{ij_1 j_2 \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_n, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \\
& \times \underline{f}_{j_1 i_1}^{\dagger(1)}(\omega_1, \mathbf{k}_1, \mathbf{p}_1) \underline{f}_{j_2 i_2}^{\dagger(1)}(\omega_2, \mathbf{k}_2, \mathbf{p}_2) \dots \underline{f}_{j_n i_n}^{\dagger(1)}(\omega_n, \mathbf{k}_n, \mathbf{p}_n)] \quad (31)
\end{aligned}$$

$$\begin{aligned}
& \zeta_{ii_1 i_2 \dots i_n}^{(n)}(t_1, t_2, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n) = \\
& \Theta(t_1) \Theta(t_2) \dots \Theta(t_n) \int_0^\infty d\omega_1 \int_0^d \omega_2 \dots \int_0^\infty d\omega_n \frac{\sin \omega_1 t_1}{\omega_1} \frac{\sin \omega_2 t_2}{\omega_2} \dots \frac{\sin \omega_n t_n}{\omega_n} \\
& \times \int d^3 p_1 \int d^3 p_2 \dots \int d^3 p_n [\underline{g}_{ij_1 j_2 \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_n, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \\
& \times \underline{g}_{j_1 i_1}^{\dagger(1)}(\omega_1, \mathbf{k}_1, \mathbf{p}_1) \underline{g}_{j_2 i_2}^{\dagger(1)}(\omega_2, \mathbf{k}_2, \mathbf{p}_2) \dots \underline{g}_{j_n i_n}^{\dagger(1)}(\omega_n, \mathbf{k}_n, \mathbf{p}_n)] \quad (32)
\end{aligned}$$

where $\Theta(t)$ is the step function and over the repeated indices j_1, j_2, \dots, j_n should be summed. There are some symmetry conditions that the electric and magnetic susceptibility tensors of various rank should satisfy. The tensors $\chi^{(n)}$ and $\zeta^{(n)}$ may satisfy the symmetry conditions [21]

$$\begin{aligned}
& \chi_{ii_1 i_2 \dots i_\alpha \dots i_\beta \dots i_n}^{(n)}(t_1, t_2, \dots, t_\alpha, \dots, t_\beta, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_n) \\
& = \chi_{ii_1 i_2 \dots i_\beta \dots i_\alpha \dots i_n}^{(n)}(t_1, t_2, \dots, t_\beta, \dots, t_\alpha, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_n) \quad (33)
\end{aligned}$$

$$\begin{aligned}
& \zeta_{ii_1 i_2 \dots i_\alpha \dots i_\beta \dots i_n}^{(n)}(t_1, t_2, \dots, t_\alpha, \dots, t_\beta, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_n) \\
& = \zeta_{ii_1 i_2 \dots i_\beta \dots i_\alpha \dots i_n}^{(n)}(t_1, t_2, \dots, t_\beta, \dots, t_\alpha, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_n) \quad (34)
\end{aligned}$$

From the definitions (31) and (32) it is clear that the symmetry relations (33) and (34) are satisfied provided that we impose the conditions

$$\begin{aligned}
& \underline{f}_{ij_1 j_2 \dots j_\alpha \dots j_\beta \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_\alpha, \dots, \omega_\beta, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_n) \\
& = \underline{f}_{ij_1 j_2 \dots j_\beta \dots j_\alpha \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_\beta, \dots, \omega_\alpha, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_n) \quad (35)
\end{aligned}$$

$$\begin{aligned}
& \underline{g}_{ij_1 j_2 \dots j_\alpha \dots j_\beta \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_\alpha, \dots, \omega_\beta, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_n) \\
&= \underline{g}_{ij_1 j_2 \dots j_\beta \dots j_\alpha \dots j_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_\beta, \dots, \omega_\alpha, \dots, \omega_n, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_\beta, \dots, \mathbf{k}_\alpha, \dots, \mathbf{k}_n)
\end{aligned} \tag{36}$$

on the coupling tensors $\underline{\mathbf{f}}^{(n)}$ and $\underline{\mathbf{g}}^{(n)}$ for $n = 2, 3, \dots$

In constitutive relations (29) and (30) $\underline{\mathbf{P}}_N(\mathbf{k}, t)$ and $\underline{\mathbf{M}}_N(\mathbf{k}, t)$ are the noise polarization and magnetization fields, respectively, where their components are written in terms of the $\underline{\mathbf{X}}_{N\omega}(\mathbf{k}, t)$ and $\underline{\mathbf{Y}}_{N\omega}(\mathbf{k}, t)$ as

$$\begin{aligned}
P_{Ni}(\mathbf{k}, t) &= \int_0^\infty d\omega_1 \int d^3 k_1 \underline{f}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{X}_{N\omega_1}^j(\mathbf{k}_1, t) \\
&+ \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3 k_1 \int d^3 k_2 [\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{X}_{N\omega_1}^j(\mathbf{k}_1, t) \underline{X}_{N\omega_2}^k(\mathbf{k}_2, t)] \\
&\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3 k_1 \int d^3 k_2 \left[\underline{f}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \int_0^t dt' \frac{\sin \omega_2(t-t')}{\omega_2} \right. \\
&\times \left. \int d^3 p_2 \underline{f}_{kl}^{\dagger(1)}(\omega_2, \mathbf{p}_2, \mathbf{k}_2) (\underline{X}_{N\omega_1}^j(\mathbf{k}_1, t) \underline{E}^l(\mathbf{p}_2, t') + \underline{E}^l(\mathbf{p}_2, t') \underline{X}_{N\omega_1}^j(\mathbf{k}_1, t)) \right] + \dots
\end{aligned} \tag{37}$$

$$\begin{aligned}
M_{Ni}(\mathbf{k}, t) &= \int_0^\infty d\omega_1 \int d^3 k_1 \underline{g}_{ij}^{(1)}(\omega_1, \mathbf{k}, \mathbf{k}_1) \underline{Y}_{N\omega_1}^j(\mathbf{k}_1, t) \\
&+ \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3 k_1 \int d^3 k_2 [\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \underline{Y}_{N\omega_1}^j(\mathbf{k}_1, t) \underline{Y}_{N\omega_2}^k(\mathbf{k}_2, t)] \\
&\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int d^3 k_1 \int d^3 k_2 \left[\underline{g}_{ijk}^{(2)}(\omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \int_0^t dt' \frac{\sin \omega_2(t-t')}{\omega_2} \right. \\
&\times \left. \int d^3 p_2 \underline{g}_{kl}^{\dagger(1)}(\omega_2, \mathbf{p}_2, \mathbf{k}_2) (\underline{Y}_{N\omega_1}^j(\mathbf{k}_1, t) \underline{B}^l(\mathbf{p}_2, t') + \underline{B}^l(\mathbf{p}_2, t') \underline{Y}_{N\omega_1}^j(\mathbf{k}_1, t)) \right] + \dots
\end{aligned} \tag{38}$$

where the symmetry conditions (35) and (36) have been used, the summation should be done on the repeated indices and the three points ... indicate the terms containing the coupling tensors more than the third rank.

6 The time dependence of the electric and magnetic field

Let we write the the electric field $\underline{\mathbf{E}}(\mathbf{k}, t)$ in terms of its temporal Fourier transform as

$$\underline{\mathbf{E}}(\mathbf{k}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d\omega \tilde{\underline{\mathbf{E}}}(\mathbf{k}, \omega) e^{i\omega t} \quad (39)$$

then, using this relation and similar transformations for the other fields and combination of the Maxwell equations $i\mathbf{k} \times \underline{\mathbf{E}}(\mathbf{k}, t) = \underline{\dot{\mathbf{B}}}(\mathbf{k}, t)$, $\underline{\dot{\mathbf{D}}}(\mathbf{k}, t) = i\mathbf{k} \times \underline{\mathbf{H}}(\mathbf{k}, t)$ give us

$$\mathbf{k} \times \mathbf{k} \times \tilde{\underline{\mathbf{E}}}(\mathbf{k}, \omega) + \frac{\omega^2}{c^2} \tilde{\underline{\mathbf{E}}}(\mathbf{k}, \omega) = -\mu_0 \omega^2 \tilde{\underline{\mathbf{P}}}(\mathbf{k}, \omega) - \mu_0 \omega \mathbf{k} \times \tilde{\underline{\mathbf{M}}}(\mathbf{k}, \omega) \quad (40)$$

for the electric field in frequency domain where $\tilde{\underline{\mathbf{P}}}(\mathbf{k}, \omega)$ and $\tilde{\underline{\mathbf{M}}}(\mathbf{k}, \omega)$ are the temporal Fourier transforms of $\underline{\mathbf{P}}(\mathbf{k}, t)$ and $\underline{\mathbf{M}}(\mathbf{k}, t)$, respectively. Now substituting $\tilde{\underline{\mathbf{P}}}(\mathbf{k}, \omega)$ and $\tilde{\underline{\mathbf{M}}}(\mathbf{k}, \omega)$, using the constitutive relations (29) and (30), in (40) leads to the integral equation

$$\begin{aligned} & \Lambda_{ii_1}(\mathbf{k}, \omega) \tilde{\underline{E}}_{i_1}(\mathbf{k}, \omega) + (2\pi)^{\frac{3}{2}} \mu_0 \omega^2 \int d^3 k_1 \tilde{\chi}_{ii_1}^{(1)}(\omega, \mathbf{k}, \mathbf{k}_1) \tilde{\underline{E}}_{i_1}(\mathbf{k}_1, \omega) \\ & + (2\pi)^{\frac{3}{2}} \mu_0 \omega^2 \int d^3 k_1 \int d^3 k_2 \int_0^\infty d\omega_1 \\ & \times [\tilde{\chi}_{ii_1 i_2}^{(2)}(\omega_1, \omega - \omega_1, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \tilde{\underline{E}}_{i_1}(\mathbf{k}_1, \omega_1) \tilde{\underline{E}}_{i_2}(\mathbf{k}_2, \omega - \omega_1)] \\ & + \dots \\ & + (2\pi)^{\frac{3}{2}} \mu_0 \omega \int d^3 k_1 \epsilon_{ij_1 j_2} k_{j_1} \tilde{\zeta}_{j_2 i_1}^{(1)}(\omega, \mathbf{k}, \mathbf{k}_1) \frac{(\mathbf{k}_1 \times \tilde{\underline{E}}(\mathbf{k}_1, \omega))_{i_1}}{-\omega} \\ & + (2\pi)^{\frac{3}{2}} \mu_0 \omega \int d^3 k_1 \int d^3 k_2 \int_0^\infty d\omega_1 \\ & \times \left[\epsilon_{ij_1 j_2} k_{j_1} \tilde{\zeta}_{j_2 i_1 i_2}^{(2)}(\omega_1, \omega - \omega_1, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \right. \\ & \times \left. \frac{(\mathbf{k}_1 \times \tilde{\underline{E}}(\mathbf{k}_1, \omega_1))_{i_1}}{-\omega_1} \frac{(\mathbf{k}_2 \times \tilde{\underline{E}}(\mathbf{k}_2, \omega - \omega_1))_{i_2}}{-(\omega - \omega_1)} \right] + \dots = \tilde{\underline{J}}_i(\mathbf{k}, \omega) \quad (41) \end{aligned}$$

for the electric field where the Maxwell equation $i\mathbf{k} \times \underline{\mathbf{E}}(\mathbf{k}, t) = \dot{\underline{\mathbf{B}}}(\mathbf{k}, t)$ has been used, ϵ_{ijk} is three dimensional levi-Civita symbol and

$$\begin{aligned}\Lambda_{ii_1}(\mathbf{k}, \omega) &= \epsilon_{ijm}\epsilon_{mni_1}k_jk_n + \frac{\omega^2}{c^2} \\ \tilde{\underline{\mathbf{J}}}_i(\mathbf{k}, \omega) &= -\mu_0\omega^2\tilde{\underline{P}}_{Ni}(\mathbf{k}, \omega) - \mu_0\omega\epsilon_{ijl}k_j\tilde{\underline{M}}_{Nl}(\mathbf{k}, \omega)\end{aligned}\quad (42)$$

In integral equation (41) $\tilde{\chi}^{(n)}, \tilde{\zeta}^{(n)}, n = 1, 2, \dots$ are the temporal Fourier transforms of the Susceptibility tensors $\chi^{(n)}, \zeta^{(n)}$ which is defined as

$$\begin{aligned}\chi_{ii_1i_2\dots i_n}^{(n)}(t_1, t_2, \dots, t_n, \mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n) &= \frac{1}{(2\pi)^{\frac{3(n+1)}{2}}} \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \dots \int_{-\infty}^{+\infty} d\omega_n \\ &\times [\chi_{ii_1\dots i_n}^{(n)}(\omega_1, \omega_2, \dots, \omega_n, \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n) e^{i\omega_1 t_1 + i\omega_2 t_2 + \dots + i\omega_n t_n}]\end{aligned}\quad (43)$$

and $\tilde{\zeta}^{(n)}$ is defined in a similar way.

The equation (41) is an integral equation with kernels $\tilde{\chi}^{(n)}$ and $\tilde{\zeta}^{(n)}$ and the source term $\tilde{\underline{\mathbf{J}}}(\mathbf{k}, \omega)$. As well, from the definitions of the noise polarization and magnetization densities given by (37) and (38), it is seen that the source term $\tilde{\underline{\mathbf{J}}}(\mathbf{k}, \omega)$ is dependent on the electric and magnetic fields. The integral equations such as (41) can be solved by an iteration method[22]. In the zero order approximation we neglect the integrals in the left hand of the equation(41) and write the components of the electric field $\tilde{\underline{\mathbf{E}}}(\mathbf{k}, \omega)$ as follows

$$\tilde{\underline{E}}_i^{(0)}(\mathbf{k}, \omega) = \Lambda_{ij}^{-1}(\mathbf{k}, \omega) \tilde{\underline{J}}_j^{(0)}(\mathbf{k}, \omega) \quad (44)$$

where $\tilde{\underline{\mathbf{J}}}^{(0)}(\mathbf{k}, \omega)$ is the source term $\tilde{\underline{\mathbf{J}}}(\mathbf{k}, \omega)$ given by (42) from which the terms containing the electric and magnetic fields have been eliminated. Therefore in the zero order approximation the time dependence of the electric and magnetic fields can be written as

$$\begin{aligned}\underline{\mathbf{E}}^{(0)}(\mathbf{k}, t) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d\omega \tilde{\underline{\mathbf{E}}}^{(0)}(\mathbf{k}, \omega) e^{i\omega t} \\ \underline{\mathbf{B}}^{(0)}(\mathbf{k}, t) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d\omega \frac{\mathbf{k} \times \tilde{\underline{\mathbf{E}}}^{(0)}(\mathbf{k}, \omega)}{-\omega} e^{i\omega t}\end{aligned}\quad (45)$$

In n -th order approximation the components of the temporal Fourier transform of the electric field for $n = 1, 2, 3, \dots$ can be computed using the following recurrence relation

$$\begin{aligned}
\tilde{E}_i^{(n)}(\mathbf{k}, \omega) = & -(2\pi)^{\frac{3}{2}} \mu_0 \omega^2 \int d^3 k_1 \left[\Lambda_{ij}^{(-1)}(\mathbf{k}, \omega) \tilde{\chi}_{ji_1}^{(1)}(\omega, \mathbf{k}, \mathbf{k}_1) \tilde{E}_{i_1}^{(n-1)}(\mathbf{k}_1, \omega) \right] \\
& - (2\pi)^{\frac{3}{2}} \mu_0 \omega^2 \int d^3 k_1 \int d^3 k_2 \int_0^\infty d\omega_1 \\
& \times \left[\Lambda_{ij}^{(-1)}(\mathbf{k}, \omega) \tilde{\chi}_{ji_1 i_2}^{(2)}(\omega_1, \omega - \omega_1, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \tilde{E}_{i_1}^{(n-1)}(\mathbf{k}_1, \omega_1) \tilde{E}_{i_2}^{(n-1)}(\mathbf{k}_2, \omega - \omega_1) \right] + \dots \\
& - (2\pi)^{\frac{3}{2}} \mu_0 \omega \int d^3 k_1 \left[\Lambda_{ij}^{(-1)}(\mathbf{k}, \omega) \epsilon_{jj_1 j_2} k_{j_1} \tilde{\zeta}_{j_2 i_1}^{(1)}(\omega, \mathbf{k}, \mathbf{k}_1) \frac{[\mathbf{k}_1 \times \tilde{E}^{(n-1)}(\mathbf{k}_1, \omega)]_{i_1}}{-\omega} \right] \\
& - (2\pi)^{\frac{3}{2}} \mu_0 \omega \int d^3 k_1 \int d^3 k_2 \int_0^\infty d\omega_1 \\
& \times \left[\Lambda_{ij}^{(-1)}(\mathbf{k}, \omega) \epsilon_{jj_1 j_2} k_{j_1} \tilde{\zeta}_{j_2 i_1 i_2}^{(2)}(\omega_1, \omega - \omega_1, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \right. \\
& \times \left. \frac{[\mathbf{k}_1 \times \tilde{E}^{(n-1)}(\mathbf{k}_1, \omega_1)]_{i_1}}{-\omega_1} \frac{[\mathbf{k}_2 \times \tilde{E}^{(n-1)}(\mathbf{k}_2, \omega - \omega_1)]_{i_2}}{-(\omega - \omega_1)} \right] + \dots \\
& + \Lambda_{ij}^{(-1)}(\mathbf{k}, \omega) \tilde{J}_j^{(n-1)}(\mathbf{k}, \omega)
\end{aligned} \tag{46}$$

where the three points ... indicate the terms containing the electric and magnetic susceptibility tensors more than the third rank and the components of $\tilde{\mathbf{J}}^{(n-1)}(\mathbf{k}, \omega)$ are defined as

$$\tilde{J}_i^{(n-1)}(\mathbf{k}, \omega) = -\mu_0 \omega^2 \tilde{P}_{Ni}^{(n-1)}(\mathbf{k}, \omega) - \mu_0 \omega \epsilon_{ijl} k_j \tilde{M}_{Nl}^{(n-1)}(\mathbf{k}, \omega) \tag{47}$$

and $\tilde{\mathbf{P}}_N^{(n-1)}(\mathbf{k}, \omega)$ and $\tilde{\mathbf{M}}_N^{(n-1)}(\mathbf{k}, \omega)$ are the temporal Fourier transforms of the noise polarization and magnetization fields given by (37) and (38) in which the electric and magnetic fields $\mathbf{E}^{(n-1)}(\mathbf{k}, t)$, $\mathbf{B}^{(n-1)}(\mathbf{k}, t)$ are inserted in those terms of $\mathbf{P}_N(\mathbf{k}, t)$, $\mathbf{M}_N(\mathbf{k}, t)$ that are dependent on the electric and magnetic fields.

7 Conclusion

By introducing a Lagrangian for the total system, that is the nonlinear anisotropic magnetodielectric medium plus the electromagnetic field, a canon-

ical quantization was introduced for both the medium and the electromagnetic field. The coupling tensors appeared in the Lagrangian of the total system had a crucial role in this quantization scheme. So that the electric and magnetic polarization fields could be written in terms of the coupling tensors and the space dependent three dimensional harmonic oscillators modeling the medium. As well the electric and magnetic susceptibility tensors of the medium were obtained in terms of the coupling tensors. Also the constitutive relations of the magnetodielectric medium successfully could be obtained using the Heisenberg equations of the harmonic oscillators modeling the medium.

References

- [1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet.51,793(1948).
- [2] P. W. Milonni, *The quantum vacuum: An introduction to quantum electrodynamics* (Academic Press, San Diego,1994).
- [3] H. Casimir, D. Polder, Phys. Rev. A 73, 360(1948).
- [4] E. Lifshitz, *The theory of molecular attractive forces between-solids*(1956).
- [5] I. E. e. Dzyaloshinskii, E. Lifshitz, L. P. Pitaevskii, *General theory of Van der waals forces*, Physics-Uspekhi 4,153-176(1961).
- [6] C. Raabe, D. G. Welsch, Phys. Rev. A 71, 013814 (2005).
- [7] R. Matloob, Phys. Rev. A 70, 062110(2004).
- [8] R. Matloob, Phys. Rev. A 60, 3421(1999).
- [9] R. Podgornik, P. L. Hansen, V. A. Parsegian, J. Chem. Phys. Vol. 119, No. 2(2003).
- [10] R. Golestanian, Phys. Rev. Lett. 95, 230601 (2005).
- [11] E. M. purcell, Phys. Rev. 69, 681 (1946).
- [12] E. Yablonovitch, Phys. Rev. Lett. 58,2059(1987).

- [13] S. M. Barnett, B. Huttner, R. Loudon, Phys. Rev. Lett. 68, 3698 (1992).
- [14] S. M. Barnett, B. Huttner, R. Loudon, R. Matloob, J. Phys. B 29, 3763 (1996).
- [15] B. Huttner, S. M. Barnett, Phys. Rev. A 46, 4306 (1992)
- [16] H. T. Dung, L. Knöll, D. G. Welsch, Phys. Rev. A 57, 3931 (1998).
- [17] F. Kheirandish, M. Amooshahi, M. Soltani, J. Phys. B: At. Mol. Opt. Phys. 42, 075504 (2009).
- [18] M. Amooshahi, J. Math. Phys. 50, 062301 (2009)
- [19] M. Amooshahi, Eur. Phys. J. D 69 (2015)
- [20] M. Amooshahi, E. Amooghorban, Annals of physics, 325, 1976-1986 (2010)
- [21] Guang S. He, Song H. Liu, *Physics of nonlinear optics*, World Scientific (1999).
- [22] Sadri Hassani, *Foundations of Mathematical Physics*, McGraw-Hill (1991)